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Relationship among continuity conditions and null-additivity conditions in non-additive measure theory *

朝比奈 伸, 内野 健太, 室伏 俊明
Shin Asahina, Uchino Kenta, Toshiaki Murofushi

東京工業大学大学院 総合理工学研究科 知能システム科学専攻
Department of Computational Intelligence and Systems Science,
Tokyo Institute of Technology

Abstract

The paper shows all the implication relationship among six continuity conditions and two null-additivity conditions with respect to non-additive measures. The six continuity conditions are continuity from above, order continuity, strong order continuity, exhaustivity, continuity from below, and null-continuity, and the two null-additivity conditions are null-additivity and weak null-additivity. Furthermore the implication relationship is summarized in a diagram.

1 Introduction

So far, non-additive measure theory has been constructed along lines of the classical measure theory [1, 12]; here a non-additive measure means a non-negative monotone set function. In general, theorems in the classical measure theory do not hold in non-additive measure theory without additional conditions. Therefore several conditions with respect to non-additive measures have been given in order that such theorems hold [2, 3, 6, 7, 8, 11, 12]. For example, Egoroff's theorem, which is one of the most important convergence theorems in the classical measure theory, does not hold in non-additive measure theory, but it holds under continuity from above and below, that is, the continuity from above and below is a sufficient condition for the theorem [7]. In the same way, sufficient conditions for other theorems have been given [2, 3, 6, 7, 8, 11, 12], but the relationship among these conditions have hardly been studied so far. For this reason, it is very important to clarify the relationship among them.

In this paper, we investigate eight well-known conditions with respect to continuity and null-additivity, and show all the implication relationship among them.

This paper is organized as follows: In Section 2, we give the definitions of a non-additive measure and the eight conditions discussed in this paper. In Section 3, the main results are stated. We give all the implication relations among the eight conditions, and show through ten counterexamples that there exist no further implication relations. In Section 4, we state the conclusion of this paper, and give a subject of future research.

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2 Notation and definitions

Let X be a non-empty set, $\mathfrak{F}$ be a σ-algebra of subsets of X, and let $\mathbb{N}$ denote the set of all positive integers.

According to Murofushi [8], we define a non-additive measure as follows:

Definition 1 A *non-additive measure* on $\mathfrak{F}$ is a set function $\mu : \mathfrak{F} \to [0, \infty]$ satisfying the following two conditions:

(i) $\mu(\emptyset) = 0$,

(ii) $A, B \in \mathfrak{F}, A \subset B \Rightarrow \mu(A) \leq \mu(B)$.

A non-additive measure μ is said to be *finite* if $\mu(X) < \infty$.

Unless stated otherwise, all subsets are supposed to belong to $\mathfrak{F}$ and μ is assumed to be a non-additive measure on $\mathfrak{F}$.

The following conditions concerning null-additivity are defined by Wang and Klir.

Definition 2 (i) μ is said to be *null-additive* if $\mu(A \cup B) = \mu(A)$ for every set B such that $\mu(B) = 0$ [11].

(ii) μ is said to be *weakly null-additive* if $\mu(A \cup B) = 0$ for every sets A and B such that $\mu(A) = \mu(B) = 0$ [12].

The following conditions concerning continuity are defined by Drewnowski [4], Jiang et al. [5] and, Uchino and Murofushi [9].

Definition 3 (i) μ is said to be *continuous from above* [resp. *below*] if

$$\mu\left(\lim_{n\to\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$$

for every decreasing [resp. increasing] sequence $\{A_n\}$. In addition, μ is said to be *continuous* if it is continuous both from above and below.

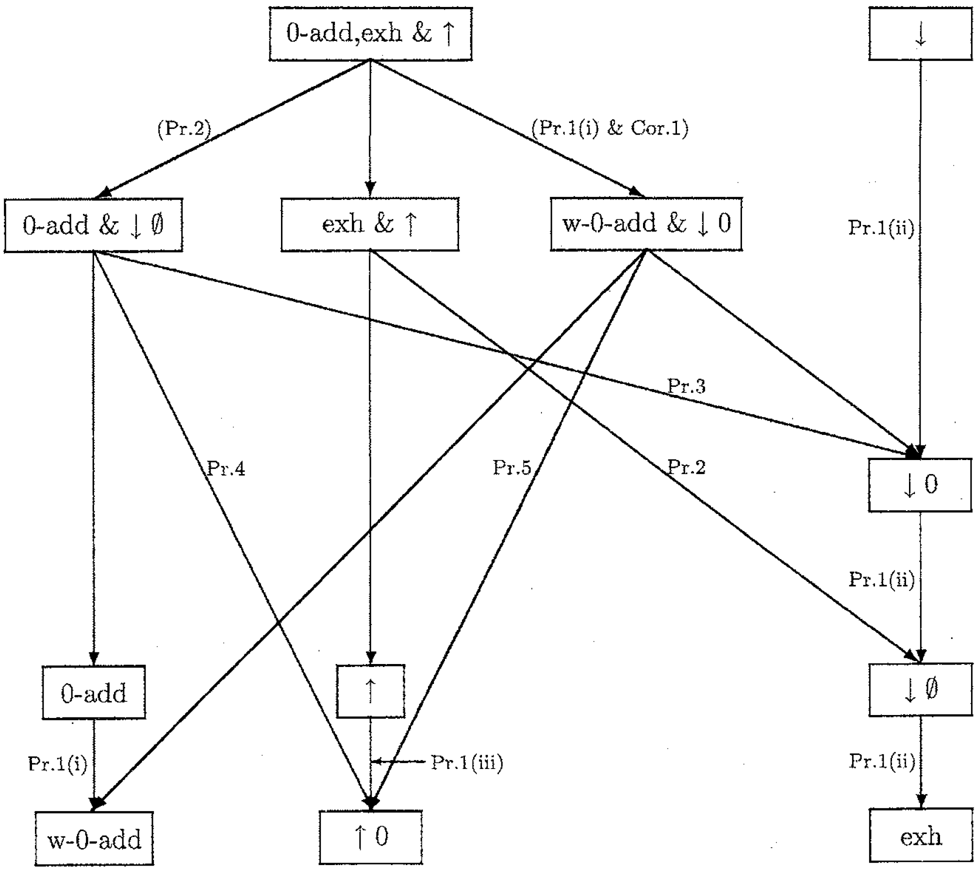
(ii) μ is said to be *order continuous* if it is continuous at the empty set, that is, $\lim_{n\to\infty} \mu(A_n) = 0$ for every decreasing sequence $\{A_n\}$ such that $A_n \downarrow \emptyset$ [4].

(iii) μ is said to be *strongly order continuous* if it is continuous at measurable sets of measure zero, that is, $\lim_{n\to\infty} \mu(A_n) = 0$ for every decreasing sequence $\{A_n\}$ such that $\mu(\bigcap_{n=1}^{\infty} A_n) = 0$ [5].

(iv) μ is said to be *null-continuous* if $\mu(\bigcup_{n=1}^{\infty} A_n) = 0$ for every increasing sequence $\{A_n\}$ such that $\mu(A_n) = 0$ for every n [9].

(v) μ is said to be *exhaustive* if $\lim_{n\to\infty} \mu(A_n) = 0$ for every disjoint sequence $\{A_n\}$ [4].

Note that every classical finite measure satisfies all the conditions stated above.



0-add : null-additivity
$\downarrow$: continuity from above
$\downarrow \emptyset$: order continuity
$\uparrow$: continuity from below

w-0-add : weak null-additivity
$\downarrow 0$: strong order continuity
exh : exhaustivity
$\uparrow 0$: null-continuity

Figure 1: Implication relationship

3 All the implication relationship among the eight conditions

In this section, we show all the implication relationship among the eight conditions stated in Definitions 2 and 3. We first represent the relations among them in a diagram as Fig. 1, then we discuss the implication relations, and show through ten counterexamples that there exist no other implication relations.

Figure 1 summarizes all the relations. In this diagram, an arrow from A to B means that condition A implies condition B, and its label "Pr. X" indicates the proposition showing it; for example, the fact that continuity from above implies strong order continuity is shown by Proposition 1 (ii). The arrow corresponding to a trivial relation has no label. On the other hand, the absence of a directed path means that A does not imply B; for example, null-continuity does not imply continuity from below, and continuity from below does not imply null-additivity.

3.1 Implication Relationship

We first state all the implication relations from one condition to another; most of these implication relations are obvious.

Proposition 1 (i) *If μ is null-additive, then it is weakly null-additive.*

(ii) a) *If μ is continuous from above, then it is strongly order continuous,*
b) *if μ is strongly order continuous, then it is order continuous,*
c) *if μ is order continuous, then it is exhaustive* [4].

(iii) *If μ is continuous from below, then it is null-continuous.*

Then we state the implication relations from more than one condition to another.

Proposition 2 *If μ is exhaustive and continuous from below, then it is order continuous* [3].

Proposition 3 *If μ is null-additive and order continuous, then it is strongly order continuous* [6].

By Propositions 2 and 3, we have the following corollary.

Corollary 1 *If μ is null-additive, exhaustive and continuous from below, then it is strongly order continuous.*

Proposition 4 *If μ is null-additive and order continuous, then it is null-continuous.* [10]

The following proposition is our new result.

Proposition 5 *If μ is weakly null-additive and strongly order continuous, then it is null-continuous.*

Proof. Assume that μ is weakly null-additive and strongly order continuous. Let $N_n \uparrow N$ and $\mu(N_n) = 0$ for all $n \geq 1$. We first define a subsequence $\{N_{n_m}\}$ of $\{N_n\}$ recursively as follows. Let $n_1 = 1$. For $m \geq 1$, since $\mu(N_{n_m}) = 0$ and $N_{n_m} \cup (N \setminus N_n) \downarrow N_{n_m}$ as $n \to \infty$, by strong order continuity we can choose $n_{(m+1)}$ so that $n_{(m+1)} > n_m$ and

$$\mu\left(N_{n_m} \cup (N \setminus N_{n_{(m+1)}})\right) < \frac{1}{m}.$$

Then we define

$$A = \bigcup_{i=1}^{\infty} \left(N_{n_{2i}} \setminus N_{n_{(2i-1)}}\right), \qquad B = N \setminus A.$$

For every i, since $A \subset N_{n_{2i}} \cup (N \setminus N_{n_{(2i+1)}})$, it follows that

$$\mu(A) \leq \mu\left(N_{n_{2i}} \cup (N \setminus N_{n_{(2i+1)}})\right) < \frac{1}{2i}.$$

Hence $\mu(A) = 0$. Similarly we have $\mu(B) = 0$. Then weak null-additivity implies that

$$\mu(N) = \mu(A \cup B) = 0.$$

Therefore μ is null-continuous. $\square$

There exist no implication relations except stated above. We show it through ten counterexamples given in the following subsection.

Table 1: Summary of the examples

Example	0-add	w-0-add	$\downarrow$	$\downarrow 0$	$\downarrow \emptyset$	exh	$\uparrow$	$\uparrow 0$
Ex. 1	F	F	T	T	T	T	T	T
Ex. 2	F	T	T	T	T	T	T	T
Ex. 3	T	T	F	F	F	F	T	T
Ex. 4	T	T	F	F	F	T	F	T
Ex. 5	F	T	F	F	T	T	T	T
Ex. 6	T	T	F	T	T	T	T	T
Ex. 7	F	F	T	T	T	T	F	F
Ex. 8	F	T	F	F	T	T	F	F
Ex. 9	T	T	F	F	F	T	F	F
Ex. 10	T	T	T	T	T	T	F	T

0-add : null-additivity
$\downarrow$: continuity from above
$\downarrow \emptyset$: order continuity
$\uparrow$: continuity from below
w-0-add : weak null-additivity
$\downarrow 0$: strong order continuity
exh : exhaustivity
$\uparrow 0$: null-continuity

3.2 Examples with respect to non-additive measures

The absence of other arrows in Fig. 1 is shown by counterexamples, Examples 1–10, in this subsection. Table 1 summarizes all the examples. In this table, the symbol "T" in the cell at the row of "Ex. A" and the column of a condition "C" means that the non-additive measure in Example A satisfies the condition C, and the symbol "F" means that it does not; for instance, the non-additive measure in Example 6 satisfies all conditions except continuity from above.

In the following two examples, non-additive measures are defined on a finite set. Obviously such non-additive measures satisfy all the continuity conditions.

Example 1 Let $X = \{0, 1\}$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset, \{0\} \text{ or } \{1\}, \\ 1 & \text{if } A = X. \end{cases}$$

Then μ is not weakly null-additive; indeed, if we take $A = \{0\}$ and $B = \{1\}$, then $\mu(A) = \mu(B) = 0$, but we have $\mu(A \cup B) = \mu(X) = 1$ by the definition of μ. It follows from this that it is not null-additive either. On the other hand, obviously μ satisfies the other conditions.

Example 2 Let $X = \{0, 1\}$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset \text{ or } \{0\}, \\ 1 & \text{if } A = \{1\}, \\ 2 & \text{if } A = X. \end{cases}$$

Then μ is not null-additive; indeed, if we take A and B as in Example 1, then $\mu(A) = 0$, but we have $\mu(A \cup B) = \mu(X) = 2 \neq 1 = \mu(B)$ by the definition of μ. On the other hand, obviously μ satisfies the other conditions.

In the remaining examples, non-additive measures are defined on a countable set.

Example 3 Let $X = \mathbb{N}$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A = \emptyset, \\ 1 & \text{if } A \neq \emptyset. \end{cases}$$

Then μ is not exhaustive. Indeed if we take $A_n = \{n\}$ for $n = 1, 2, \ldots$, then $\{A_n\}$ is a disjoint sequence and $\mu(A_n) = 1$ for every n by the definition of μ. Hence we have $\lim_{n\to\infty} \mu(A_n) = 1$. Thus μ is not exhaustive. It follows from this that it does not satisfy the following three conditions: continuity from above, strong order continuity and order continuity. On the other hand, obviously μ satisfies the other conditions.

Example 4 Let $X = \mathbb{N}$, $m(A) = \sum_{n\in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} m(A) & \text{if } |A^c| = \infty, \\ 1 & \text{if } |A^c| < \infty. \end{cases}$$

Then, μ is not order continuous; indeed, if we take $A_n = \{n, n+1, \ldots\}$ for $n = 1, 2, \ldots$, then $A_n \downarrow \emptyset$ as $n \to \infty$ and $|A_n{}^c| < \infty$ for any n, but we have $\lim_{n\to\infty} \mu(A_n) = 1$ by the definition of μ. It follows from this that it is neither continuous from above nor strongly order continuous. On the other hand, we can show that μ is exhaustive. Let $\{A_n\}$ be a disjoint sequence. If $\mu(A_n) < 1$ for every n, then $\mu(A_n) = m(A_n)$ by the definition of μ. Thus $\lim_{n\to\infty} \mu(A_n) = 0$ since m is a finite measure. If not, that is, there exits a positive integer l such that $\mu(A_l) = 1$, then $|A_l{}^c| < \infty$ by the definition of μ, and since $\{A_n\}$ is disjoint, we have $A_n \subset A_l{}^c$ for any n such that $n \neq l$. Hence $\mu(A_n) = m(A_n)$ by $|A_n{}^c| = \infty$ for any $n \neq l$. Then, similarly we have $\lim_{n\to\infty} \mu(A_n) = 0$. Therefore μ is exhaustive. By the fact that μ is not order continuous but exhaustive, and Proposition 2, it is not continuous from below. Since m is a finite measure, obviously μ is null-continuous and null-additive, hence it is also weakly null-additive.

Example 5 Let $X = \mathbb{N}$, $m(A) = \sum_{n\in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if } A = \{1\}, \\ m(A) & \text{if } A \neq \{1\}. \end{cases}$$

Then μ is not strongly order continuous; indeed, if we take $A_n = \{1\} \cup \{n+1, n+2, \ldots\}$ for $n = 1, 2, \ldots$, and $A = \{1\}$, then $A_n \downarrow A$ as $n \to \infty$ and $\mu(A) = 0$, but we have $\lim_{n\to\infty} \mu(A_n) = 1/2$ by the definition of μ. It follows from this that it is not continuous from above either. On the other hand, obviously μ is weakly null-additive. Furthermore since m is a finite measure, μ is clearly order continuous and continuous from below, thus it is also exhaustive and null-continuous. By the fact that μ is not strongly order continuous but order continuous, and Proposition 3, it is not null-additive.

Example 6 Let $X = \mathbb{N}$, $m(A) = \sum_{n\in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} m(A) & \text{if } m(A) \leq 1/2, \\ 1 & \text{if } m(A) > 1/2. \end{cases}$$

Then μ is not continuous from above; indeed, if we take A_n and A as in Example 5, then $A_n \downarrow A$ as $n \to \infty$ and $\mu(A) = 1/2$, but we have $\mu(A_n) = 1$ for every n by the definition of μ. On the other hand, since m is a finite measure, obviously μ satisfies the other conditions.

Example 7 Let $X = \mathbb{N}$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A \neq X, \\ 1 & \text{if} \quad A = X. \end{cases}$$

Then μ is not null-continuous; indeed, if we take $A_n = \{1, 2, \ldots, n\}$ for $n = 1, 2, \ldots$, then $A_n \uparrow X$ and $\mu(A_n) = 0$ for every n, but we have $\mu(X) = 1$ by the definition of μ. It follows from this that it is not continuous from below either. On the other hand, obviously μ is continuous from above. It follows from this that it satisfies strong order continuity, order continuity and exhaustivity. By the fact that μ is not null-continuous but strongly order continuous, and Proposition 5, it is not weakly null-additive. Hence it is not null-additive either.

Example 8 Let $X = \mathbb{N}$, $m(A) = \sum_{n \in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad |A| < \infty, \\ m(A) & \text{if} \quad |A| = \infty. \end{cases}$$

Then, similarly in Example 7, μ is not null-continuous. Hence it is not continuous from below either. On the other hand, obviously μ is weak null-additive and order continuous. Hence it has exhaustivity. By the fact that μ is not null-continuous but order continuous, and Proposition 4, it is not null-additive. Furthermore, similarly in Example 5, μ is not strongly order continuous. Hence it is not continuous from above either.

Example 9 Let $X = \mathbb{N}$, $\mathfrak{A}$ be a non-principal ultrafilter on X, i.e., an ultrafilter containing the cofinite filter $\{M \subset X \mid |M^c| < \infty\}$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} 0 & \text{if} \quad A \notin \mathfrak{A}, \\ 1 & \text{if} \quad A \in \mathfrak{A}. \end{cases}$$

Then, similarly in Example 7, μ is not null-continuous. Hence it is not continuous from below either. On the other hand, we can show that μ is null-additive. Let A be a set such that $\mu(A) = 0$. Then $A \notin \mathfrak{A}$ by the definition of μ. If $B \notin \mathfrak{A}$, then we have $A \cup B \notin \mathfrak{A}$ since $\mathfrak{A}$ is an ultrafilter. Thus we have $\mu(A \cup B) = 0 = \mu(B)$. If $B \in \mathfrak{A}$, then we have $A \cup B \in \mathfrak{A}$. Therefore we have $\mu(A \cup B) = 1 = \mu(B)$. Hence μ is null-additive. It follows from this that it is also weakly null-additive. By the fact that μ is not null-continuous but null-additive, and Proposition 4, it is not order continuous. Therefore it is neither strongly order continuous nor continuous from above. Furthermore we can show that μ is exhaustive. Let $\{A_n\}$ be a disjoint sequence. If $\mu(A_n) = 0$ for any n, then $\lim_{n\to\infty} \mu(A_n) = 0$. If not, that is, there exists positive integer m such that $\mu(A_m) = 1$, then we have $A_m \in \mathfrak{A}$ by the definition of μ. Since $\{A_n\}$ is disjoint and $\mathfrak{A}$ is a filter, it follows that $A_n \notin \mathfrak{A}$ for every $n \neq m$, and hence that $\mu(A_n) = 0$ for every $n \neq m$. Thus we have $\lim_{n\to\infty} \mu(A_n) = 0$. Therefore μ is exhaustive.

Example 10 Let $X = \mathbb{N}$, $m(A) = \sum_{n \in A} 1/2^n$, and μ be the non-additive measure on the power set 2^X of X defined as

$$\mu(A) = \begin{cases} m(A) & \text{if} \quad m(A) < 1, \\ 2 & \text{if} \quad m(A) = 1. \end{cases}$$

Table 2: Boolean matrix of implication relationship among eight conditions

	0-add	w-0-add	$\downarrow$	$\downarrow 0$	$\downarrow \emptyset$	exh	$\uparrow$	$\uparrow 0$
0-add	—	T (Pr.1)	F (Ex.6)	F (Ex.3)	F (Ex.4)	F (Ex.3)	F (Ex.10)	F (Ex.9)
w-0-add	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	F (Ex.4)	F (Ex.3)	F (Ex.10)	F (Ex.8)
$\downarrow$	F (Ex.2)	F (Ex.1)	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	F (Ex.10)	F (Ex.7)
$\downarrow 0$	F (Ex.2)	F (Ex.1)	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	F (Ex.10)	F (Ex.7)
$\downarrow \emptyset$	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	—	T (Pr.1)	F (Ex.10)	F (Ex.7)
exh	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	F (Ex.4)	—	F (Ex.10)	F (Ex.7)
$\uparrow$	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	F (Ex.3)	F (Ex.3)	—	T (Pr.1)
$\uparrow 0$	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	F (Ex.4)	F (Ex.3)	F (Ex.10)	—
0-add,$\downarrow$	—	T (Pr.1)	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	F (Ex.10)	T (Pr.1&4)
0-add,$\downarrow 0$	—	T (Pr.1)	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	F (Ex.10)	T (Pr.1&4)
0-add,$\downarrow \emptyset$	—	T (Pr.1)	F (Ex.6)	T (Pr.3)	—	T (Pr.1)	F (Ex.10)	T (Pr.4)
0-add,exh	—	T (Pr.1)	F (Ex.6)	F (Ex.4)	F (Ex.4)	—	F (Ex.10)	F (Ex.9)
0-add,$\uparrow$	—	T (Pr.1)	F (Ex.6)	F (Ex.3)	F (Ex.3)	F (Ex.3)	—	T (Pr.1)
0-add,$\uparrow 0$	—	T (Pr.1)	F (Ex.6)	F (Ex.3)	F (Ex.4)	F (Ex.3)	F (Ex.10)	—
w-0-add,$\downarrow$	F (Ex.2)	—	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	F (Ex.10)	T (Pr.1&5)
w-0-add,$\downarrow 0$	F (Ex.2)	—	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	F (Ex.10)	T (Pr.5)
w-0-add,$\downarrow \emptyset$	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	—	T (Pr.1)	F (Ex.10)	F (Ex.8)
w-0-add,exh	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	F (Ex.4)	—	F (Ex.10)	F (Ex.8)
w-0-add,$\uparrow$	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	F (Ex.3)	F (Ex.3)	—	T (Pr.1)
w-0-add,$\uparrow 0$	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	F (Ex.4)	F (Ex.3)	F (Ex.10)	—
$\downarrow$,$\uparrow$	F (Ex.2)	F (Ex.1)	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	—	T (Pr.1)
$\downarrow$,$\uparrow 0$	F (Ex.2)	F (Ex.1)	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	F (Ex.10)	—
$\downarrow 0$,$\uparrow$	F (Ex.2)	F (Ex.1)	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	—	T (Pr.1)
$\downarrow 0$,$\uparrow 0$	F (Ex.2)	F (Ex.1)	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	F (Ex.10)	—
$\downarrow \emptyset$,$\uparrow$	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	—	T (Pr.1)	—	T (Pr.1)
$\downarrow \emptyset$,$\uparrow 0$	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	—	T (Pr.1)	F (Ex.10)	—
exh,$\uparrow$	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	T (Pr.2)	—	—	T (Pr.1)
exh,$\uparrow 0$	F (Ex.2)	F (Ex.1)	F (Ex.6)	F (Ex.5)	F (Ex.4)	—	F (Ex.10)	—
0-add,$\downarrow$,$\uparrow$	—	T (Pr.1)	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	—	T (Pr.1)
0-add,$\downarrow$,$\uparrow 0$	—	T (Pr.1)	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	F (Ex.10)	—
0-add,$\downarrow 0$,$\uparrow$	—	T (Pr.1)	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	—	T (Pr.1)
0-add,$\downarrow 0$,$\uparrow 0$	—	T (Pr.1)	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	F (Ex.10)	—
0-add,$\downarrow \emptyset$,$\uparrow$	—	T (Pr.1)	F (Ex.6)	T (Pr.3)	—	T (Pr.1)	—	T (Pr.1)
0-add,$\downarrow \emptyset$,$\uparrow 0$	—	T (Pr.1)	F (Ex.6)	T (Pr.3)	—	T (Pr.1)	F (Ex.10)	—
0-add,exh,$\uparrow$	—	T (Pr.1)	F (Ex.6)	T (Cor.1)	T (Pr.2)	—	—	T (Pr.1)
0-add,exh,$\uparrow 0$	—	T (Pr.1)	F (Ex.6)	F (Ex.4)	F (Ex.4)	—	F (Ex.10)	—
w-0-add,$\downarrow$,$\uparrow$	F (Ex.2)	—	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	—	T (Pr.1)
w-0-add,$\downarrow$,$\uparrow 0$	F (Ex.2)	—	—	T (Pr.1)	T (Pr.1)	T (Pr.1)	F (Ex.10)	—
w-0-add,$\downarrow 0$,$\uparrow$	F (Ex.2)	—	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	—	T (Pr.1)
w-0-add,$\downarrow 0$,$\uparrow 0$	F (Ex.2)	—	F (Ex.6)	—	T (Pr.1)	T (Pr.1)	F (Ex.10)	—
w-0-add,$\downarrow \emptyset$,$\uparrow$	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	—	T (Pr.1)	—	T (Pr.1)
w-0-add,$\downarrow \emptyset$,$\uparrow 0$	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	—	T (Pr.1)	F (Ex.10)	—
w-0-add,exh,$\uparrow$	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	T (Pr.2)	—	—	T (Pr.1)
w-0-add,exh,$\uparrow 0$	F (Ex.2)	—	F (Ex.6)	F (Ex.5)	F (Ex.4)	—	F (Ex.10)	—

0-add : null-additivity
w-0-add : weak null-additivity
$\downarrow$: continuity from above
$\downarrow 0$: strong order continuity
$\downarrow \emptyset$: order continuity
exh : exhaustivity
$\uparrow$: continuity from below
$\uparrow 0$: null-continuity

Then μ is not continuous from below. Indeed, if we take A_n as in Example 7, then $A_n \uparrow X$ and $m(A_n) = 1 - 1/2^n < 1$ for any n. However we have $\lim_{n\to\infty} \mu(A_n) = 1 \neq 2 = \mu(X)$ by the definition of μ. Thus μ is not continuous from below. On the other hand, since m is a finite measure, obviously μ satisfies the other conditions.

Table 2 illustrates the implication relationship among the eight conditions. The symbol "T" in the cell at the row of a condition "A" and the column of a condition "B" means that the condition "A" implies the condition "B". When there exist more than one condition in the row, they stand for the conjunction of the conditions. "(Pr. X)" indicates the proposition showing the implication; for instance, the fact that exhaustivity and continuity from below together imply order continuity is shown by Proposition 2. The symbol "F" means that the condition "A" does not imply the condition "B", and then "(Ex. X)" indicates its counterexample. The implication relationship indicated by the symbol "—" is vacuously true. Table 2 is a minimal expression of implication relationship in the same that many other implication relations can be easily derived as a simple application of Proposition 1. For example, the sufficient conditions implied by both null-additivity and weak null-additivity are exactly those implied only by null-additivity.

4 Concluding Remarks

We have showed all the implication relationship among the eight conditions with respect to non-additive measures: continuity from above, order continuity, strong order continuity, exhaustivity, continuity from below, null-continuity, null-additivity and weak null-additivity. Furthermore we have summarized these implication relationship as in Fig. 1.

In this paper, we have dealt only with the eight conditions, but there exist many other conditions which are given as a sufficient condition in order that a theorem holds. For this reason, we are researching the implication relationship among other conditions such as auto-continuity, property S, and pseudometric generating property.

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